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SOME EFFICIENT METHODS FOR OBTAINING INFINITE SERIES SOLUTIONS OF n^{th}-ORDER LINEAR ORDINARY DIFFERENTIAL EQUATIONS

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16. Abstract

The use of the theta-operator method and generalized hypergeometric functions in obtaining solutions to n^{th}-order linear ordinary differential equations is explained. For completeness, the analysis of the differential equation to determine whether the point of expansion is an ordinary point or a regular singular point is included. The superiority of the two methods shown in this report over the standard method is demonstrated by using all three of the methods to work out several examples. Also included is a compendium of formulae and properties of the theta operator and generalized hypergeometric functions which is complete enough to make the report self-contained.

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CONTENTS

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SUMMARY

In this report, some methods of obtaining series solutions for n^{th}-order linear ordinary differential equations are presented. The required analysis of the differential equation to determine whether the point of expansion is an ordinary point or a regular singular point is shown. The standard method of obtaining series solutions is presented next and, for regular singular points, the procedures are catalogued according to the value of s, the difference between the two solutions to the indicial equation. Theorems concerning the number of series solutions to second-order linear ordinary differential equations are given. Examples are worked out for each possibility.

Next, the definition and some properties of the theta operator are shown, and this is followed by a description of the use of this operator in obtaining the indicial equation and recurrence relation for n^{th}-order differential equations. The method is then applied to the same examples that were worked out by the standard method in order to show the reduced effort involved in using the theta operator.

Finally, the generalized hypergeometric function is defined and some relevant properties described. These properties are then applied to the same examples described previously.

A short reference section is included which lists all the needed properties of generalized factorials, theta operators, and generalized hypergeometric functions for obtaining series solutions to differential equations.

INTRODUCTION

Most practical problems in science and engineering are described in terms of relations between two or more physical quantities. Such relations are often in the form of n^{th}-order linear ordinary differential equations. Solutions to such equations can rarely be expressed in closed form. Usually, the solutions must be obtain numerically. Even in those cases where closed-form solutions can be obtained, numerical methods must almost always be used in order to obtain specific "y versus x" curves or tables.

Therefore, the method of solution by power series has great utility, since the form of the solution is ideally suited for obtaining a "y versus x" table even when the closed form of the solution is not known. Furthermore, series solutions are ideal for computing machines, which would otherwise often resort to them anyway to compute transcendental functions given to them in "closed form" solutions.

The preceding list of advantages of series solutions notwithstanding, a fair amount of mathematical manipulation is usually necessary in order to obtain an explicit series solution (i.e., to obtain easily computed expressions for the coefficients of the general term in the series).

This report points out that there exist some methods for greatly reducing the amount of labor usually involved in such tasks. Sadly enough, these methods are not new, but seem to have been neglected in this application even by otherwise sophisticated and well-informed applied mathematicians.

In this report, some preliminary definitions and theorems concerning the series solutions of n^{th}-order linear ordinary differential equations of complex variables are listed. The theorems cover the existence of solutions about ordinary points and regular singular points. Following this section, the "standard" method of Frobenius is reviewed. A good treatment of this method is shown in reference 1. Next, the use of the theta operator is demonstrated. Finally, the use of generalized hypergeometric functions is described. Some illustrative examples thread through the entire treatment and are solved by each of the methods. The superiority of the theta-operator and generalized-hypergeometric-function methods over the "standard" method is made patently clear, thereby.

The following two points, which are also discussed in the relevant part of the text, should be noted:

(1) Some of the definitions and all the theorems pertain to second-order linear ordinary differential equations only.

(2) All the examples shown have two-term recurrence relations. The theta-operator method, however, can be applied to n^{th}-order and will also yield recurrence relations with an arbitrary number of terms.

It should be emphasized that neither the theta operator nor the generalized hypergeometric functions are, in any sense, an invention of the author. His contribution is in the emphasis and application of these two mathematical entities towards reducing the pain associated with obtaining series solutions to linear ordinary differential equations.

Readers already familiar with the method of Frobenius and the analysis of singular points according to the solutions to the indicial equation can go directly to the section THETA-OPERATOR METHOD (p. 20).

SYMBOLS

$(a)_n$	generalized factorial symbol, $\prod_{i=0}^{n-1} (a+i)$
${}_pF_q(z)$	generalized hypergeometric function with p numerator parameters and q denominator parameters
D	d/dz, differentiation operator
R(c)	value of real part of complex number c
s	difference between two roots of indicial equation of a second-order linear differential equation; taken so that $R(s) \geq 0$
θ	z(d/dz), theta operator

SOME DEFINITIONS AND THEOREMS FOR SECOND-ORDER EQUATIONS

The material in this section applies only to second-order differential equations.

Definitions

Standard form. - The standard form for a linear second-order ordinary homogeneous differential equation is defined to be

$$\frac{d^2w}{dz^2} + p(z)\frac{dw}{dz} + q(z)w = 0 \tag{1}$$

Ordinary point. - The point z_0 (in the finite plane) is an ordinary point of equation (1) if and only if both $p(z)$ and $q(z)$ are analytic at z_0.

Singular point. - If z_0 is not an ordinary point of equation (1), it is a singular point.

Regular singular point. - Let $P(z) = (z - z_0)p(z)$, and $Q(z) = (z - z_0)^2 q(z)$. Then the point z_0 is a regular singular point of equation (1) if and only if

(1) z_0 is not an ordinary point.

(2) Both $P(z)$ and $Q(z)$ are analytic at z_0.

Irregular singular point. - If z_0 is a singular point of the differential equation (1) but is not a regular singular point, it is an irregular singular point.

Point at infinity. - To determine the nature of the point at infinity, equation (1) is transformed by using the new independent variable $\xi = 1/z$ and examining the resulting equation at $\xi = 0$. The transformed differential equation is

$$\xi^4 \frac{d^2w}{d\xi^2} + 2\xi^3 \frac{dw}{d\xi} - \xi^2 p\left(\frac{1}{\xi}\right)\frac{dw}{d\xi} + q\left(\frac{1}{\xi}\right)w = 0$$

In standard form, this becomes

$$\frac{d^2w}{d\xi^2} + \pi(\xi)\frac{dw}{d\xi} + \kappa(\xi)w = 0 \tag{1a}$$

where

$$\pi(\xi) = \frac{2\xi - p\left(\frac{1}{\xi}\right)}{\xi^2}$$

and

$$\kappa(\xi) = \frac{q\left(\frac{1}{\xi}\right)}{\xi^4}$$

Therefore, a description of the point $z = \infty$ is determined by the analysis of the behavior of $\pi(\xi)$ and $\kappa(\xi)$ at $\xi = 0$.

As an example, examine the following hypergeometric differential equation for singular points:

$$z(1 - z)\frac{d^2w}{dz^2} + (2 - 3z)\frac{dw}{dz} - w = 0$$

In standard form, this equation is

$$\frac{d^2w}{dz^2} + \frac{2 - 3z}{z(1 - z)} \frac{dw}{dz} - \frac{1}{z(1 - z)} w = 0$$

Here, $p(z)$ is not analytic at $z = 0$ or $z = 1$ so that both points are singular points. Here $zp(z) = (2 - 3z)/(1 - z)$ is analytic at $z = 0$, and $z^2q(z) = -z/(1 - z)$ is also analytic at $z = 0$. Therefore, $z = 0$ is a regular singular point. Similarly,

$$(1 - z)p(z) = \frac{2 - 3z}{z}$$

is analytic at $z = 1$ and

$$(1 - z)^2 q(z) = -\frac{(1 - z)}{z}$$

is analytic at $z = 1$. Therefore, $z = 1$ is also a regular singular point.

To examine the point at ∞, the transformation $\xi = 1/z$ is made, the differential equation (1) is rewritten in the standard form (1a), and the point $\xi = 0$ is examined. In this case,

$$\pi(\xi) = \frac{\frac{1}{\xi - 1}}{\xi}$$

and

$$\kappa(\xi) = \frac{\frac{1}{1 - \xi}}{\xi^2}$$

so that $\xi = 0$ (and, hence, $z = \infty$) is a regular singular point.

An example of an irregular singular point is the point at ∞ of the differential equation

$$\frac{d^2w}{dz^2} + k^2w = 0, \quad k = \text{a constant}$$

The transformed equation becomes

$$\xi^4 \frac{d^2w}{d\xi^2} + 2\xi^3 \frac{dw}{d\xi} + k^2 w = 0$$

which in standard form is

$$\frac{d^2w}{d\xi^2} + \frac{2}{\xi}\frac{dw}{d\xi} + \frac{k^2}{\xi^4} w = 0$$

Clearly, $\kappa(\xi)$ fails the test for a regular singular point, and thus $z = \infty$ is an irregular singular point of the differential equation.

Theorems

The following two theorems are useful in determining the existence of series solutions for second-order linear ordinary differential equations.

Theorem 1. - If z_0 is an ordinary point of the differential equation (1), then there exists a series

$$w = \sum_{n=0}^{\infty} a_n (z - z_0)^n \tag{2}$$

which satisfies the differential equation within the circle of convergence of the series, has a nonzero radius of convergence, and has two arbitrary constants a_0 and a_1. It should be recalled that the radius of convergence R is at least the distance from the point of expansion z_0 to the nearest singularity.

Theorem 2. - If z_0 is a regular singular point of the differential equation (1), then there exists a series

$$w_1(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^{n+c_1} \tag{3}$$

which satisfies the differential equation within the region of convergence of the series and has a nonzero radius of convergence. In equation (3), c_1 is defined as follows: Let

c_1 and c_2 be the roots of the indicial equation (see eq. (7)), and let $R(c_1)$ and $R(c_2)$ be their respective real parts. Then c_1 is chosen so that $R(c_1) \geq R(c_2)$.

STANDARD METHOD OF FROBENIUS

While a knowledge of the usual method is not needed for a discussion of the two nonstandard methods described in the next two sections, there are, nevertheless, two useful purposes to be served by reviewing this method before introducing the others. First, since it is already familiar to the reader, it is a pedagogically efficient way to lay the framework for series solutions to differential equations. Secondly, a proper appreciation of the difference in algebraic labor involved can only be obtained by a direct comparison of several methods of solution to the same differential equation. In the examples to be shown, therefore, the first method of solution is the method of Frobenius. The other methods are then compared with it.

Solution About an Ordinary Point

The main reason for including even a brief discussion of these solutions in this report is to emphasize the difference between solutions about an ordinary point and solutions about regular singular points. The following properties of solutions about ordinary points should be emphasized:

(1) There always exist two linearly independent series solutions about every ordinary point of a second-order linear ordinary differential equation.

(2) There is no indicial equation for solutions about an ordinary point. The solution is a power series in integral powers of $z - z_0$.

By comparison, in the case of solutions about a regular singular point, theorem 2 only guarantees that one of the two linearly independent solutions will be in the form of a power series. The second solution may also be in this form, but sometimes it cannot be expressed in a power series. Secondly, the power series solution is usually in fractional, rather than integral, powers of $z - z_0$. If $c_1 = 0$, the series solution in the case of regular singular points will also be in integral powers of $z - z_0$; but this is the exception, rather than the rule. These distinctions are referred to again in the examples.

Solution About a Regular Singular Point - The Method of Frobenius

It is convenient for this discussion to write equation (1) in the form

$$\frac{d^2w}{dz^2} + \frac{P(z)}{z - z_0}\frac{dw}{dz} + \frac{Q(z)}{(z - z_0)^2} w = 0 \tag{1b}$$

If z_0 is a regular singular point of the differential equation (1), both $P(z)$ and $Q(z)$ are analytic at z_0 and we may write

$$P(z) = \sum_{n=0}^{\infty} p_n (z - z_0)^n \qquad |z - z_0| < R_1 \tag{4}$$

$$Q(z) = \sum_{n=0}^{\infty} q_n (z - z_0)^n \qquad |z - z_0| < R_2 \tag{5}$$

where R_1 and R_2 are the radii of convergence of the two series.

The first part of the procedure consists of assuming a solution of the form

$$w = \sum_{n=0}^{\infty} a_n (z - z_0)^{n+c} \tag{6}$$

where $a_0 \neq 0$ and is otherwise arbitrary and c and the other a_n's are to be determined. The series (4), (5), and (6) are then substituted into equation (1b). The result is

$$\sum_{n=0}^{\infty} (n + c)(n + c - 1)a_n (z - z_0)^{n+c-2} + \left[\sum_{m=0}^{\infty} p_m (z - z_0)^m\right]\left[\sum_{n=0}^{\infty} (n + c)a_n (z - z_0)^{n+c-2}\right]$$
$$+ \left[\sum_{j=0}^{\infty} q_j (z - z_0)^j\right]\left[\sum_{n=0}^{\infty} a_n (z - z_0)^{n+c-2}\right] = 0$$

The lowest power of $z - z_0$ in this equation is $c - 2$. Equating the coefficient of this power to zero results in the equation

$$c(c - 1)a_0 + p_0 c a_0 + q_0 a_0 = 0$$

By hypothesis, $a_0 \neq 0$. Invoking this hypothesis results in the important indicial equation of the differential equation (1b)

$$c^2 + c(p_0 - 1) + q_0 = 0 \tag{7}$$

In order to continue the discussion, it is essential to define the quantity

$$s = c_1 - c_2 \tag{8}$$

where c_1 and c_2 are the two solutions to the indicial equation (7) and $R(c_1) \geq R(c_2)$. There are three possibilities, each governing a different procedure.

Case a: $s \neq$ an integer or zero. - In this case, two power series solutions exist, one for each solution to the indicial equation. The two solutions will therefore be of the form

$$\left.\begin{aligned} w_0 &= \sum_{n=0}^{\infty} a_n (z - z_0)^{n+c_1} \\ w_1 &= \sum_{n=0}^{\infty} b_n (z - z_0)^{n+c_2} \end{aligned}\right\} \tag{9}$$

where a_0 and b_0 are arbitrary.

Case b: $s = 0$ (or $c_1 = c_2$). - In this case only one solution to equation (1b) exists in power series form. The second linearly independent solution is of logarithmic form. Therefore, the two solutions are

$$\left.\begin{aligned} w_0 &= \sum_{n=0}^{\infty} a_n (z - z_0)^{n+c_1} \\ w_1 &= \left(\frac{\partial w_0}{\partial c}\right)_{c=c_1} \qquad \text{(a logarithmic form)} \end{aligned}\right\} \tag{10}$$

Case c: s = a positive integer. - The procedure to be followed in this case is to begin in powers of $(z - z_0)^{n+c_2}$, where c_2 is that solution to the indicial equation with the smaller real part.

The trial series is substituted into equation (1b) and, beginning with the lowest power, the coefficients of each power of $z - z_0$ are combined and then set equal to zero. In this way, some of the a_n's are determined. This procedure is continued until the coefficient a_s is reached. At this stage of the procedure, it is determined if the equation has been a "lucky" example or an "unlucky" one.

A lucky case is one in which a_s is found to be arbitrary, and two series solutions exist. On the other hand, in an "unlucky" situation, it is found that an attempt to determine a_s leads to a contradiction, that the differential equation has only one series solution, and, furthermore, that series solution which does exist is the one for the other solution to the indicial equation, so that all the work to this point has been wasted. The second solution in the unlucky case is again a logarithmic solution.

Therefore, with the aid of theorems 1 and 2 and an examination of the roots of the indicial equation, it is possible to determine in advance, except for case c, how many power series solutions exist and to estimate the difficulty involved in trying to obtain these solutions explicitly.

The following examples will serve to clarify much of the procedure described previously.

Examples

Example 1: Solution about an ordinary point. - Consider the series solution about $z = 0$ for the differential equation

$$w'' + 2zw' - 8w = 0 \tag{11}$$

This differential equation is already in standard form and $z = 0$ is an ordinary point of the given differential equation. Therefore, by theorem 1, two solutions of the form $\sum_{n=0}^{\infty} a_n z^n$ will exist and both a_0 and a_1 will be arbitrary. When $\sum_{n=0}^{\infty} a_n z_n$ is substituted into the differential equation, the result is

$$\sum_{n=0}^{\infty} n(n - 1)a_n z^{n-2} + \sum_{n=0}^{\infty} 2na_n z^n - \sum_{n=0}^{\infty} 8a_n z^n = 0$$

or

$$\sum_{n=0}^{\infty} n(n - 1)a_n z^{n-2} + \sum_{n=0}^{\infty} (2n - 8)a_n z^n = 0$$

The two lowest powers of z are z^{-2} and z^{-1}, which result in the requirements

$$0 \times a_0 = 0$$

$$0 \times a_1 = 0$$

so that both a_0 and a_1 are arbitrary, in accordance with theorem 1. Shifting index $n \to n + 2$ in the first sum, combining the two resulting sums, and equating the coefficient of z^n to zero result in the recurrence relation

$$(n + 1)(n + 2)a_{n+2} = (8 - 2n)a_n \qquad n \geq 0$$

or

$$n(n - 1)a_n = (12 - 2n)a_{n-2} \qquad n \geq 2$$

When a_0 is used as the arbitrary constant, only even powers result. When $n = 2k$, the recurrence relation becomes

$$a_{2k} = \frac{12 - 4k}{2k(2k - 1)} a_{2k-2} = \frac{2(3 - k)}{k(2k - 1)} a_{2k-2} \qquad k \geq 1$$

Therefore,

$$a_2 = \frac{2(3 - 1)}{1(2 - 1)} a_0 = 4a_0$$

$$a_4 = \frac{2(3 - 2)}{2(4 - 1)} a_2 = \frac{a_2}{3} = \frac{4}{3} a_0$$

$$a_{2k} = 0 \qquad k \geq 3$$

so that $w_0(z) = a_0\left[1 + 4z^2 + (4/3)z^4\right]$.

When a_1 is taken as the arbitrary constant, only odd powers $(n = 2k + 1)$ result. Therefore, in this case,

$$a_{2k+1} = \frac{5 - 2k}{k(2k + 1)} a_{2k-1} \qquad k \geq 1$$

This series does not terminate, and the general expression in terms of a_1 is found to be

$$a_{2k+1} = \frac{(5 - 2)(5 - 4) \ldots (5 - 2k)}{k!\left[1 \times 3 \times 5 \ldots (2k + 1)\right]} a_1$$

Therefore, the second solution (which is linearly independent of $w_0(z)$) is

$$w_1(z) = a_1 z + a_1 \sum_{k=1}^{\infty} \frac{(5 - 2)(5 - 4) \ldots (5 - 2k)}{1 \times 3 \times 5 \ldots (2k + 1)} \frac{z^{2k+1}}{k!}$$

Example 2: Solution about a regular singular point where $s \neq 0$ and $s \neq$ an integer. - Consider the series solutions about $z = 0$ of the differential equation

$$2zw'' + w' - w = 0 \tag{12}$$

It is clear that $z = 0$ is a regular singular point of this differential equation. Writing the differential equation in the form (1b) we obtain

$$w'' + \frac{\frac{1}{2}}{z} w' - \frac{\frac{z}{2}}{z^2} w = 0 \tag{12a}$$

The first requirement in a series solution about a regular singular point is the indicial equation. Here, $p_0 = 1/2$, $q_0 = 0$ so that equation (7) becomes

$$c^2 + c\left(\frac{1}{2} - 1\right) = 0$$

Therefore, $c = 0$ and $1/2$ and $s = 1/2$. This means that the differential equation belongs to case a and there will be two series solutions. When $w = \sum_{n=0}^{\infty} a_n z^{n+c}$ is substituted into equation (12), the result is

$$\sum_{n=0}^{\infty} 2(n + c)(n + c - 1)a_n z^{n+c-1} + \sum_{n=0}^{\infty} (n + c)a_n z^{n+c-1} - \sum_{n=0}^{\infty} a_n z^{n+c} = 0$$

The indicial equation has already been obtained, so it is not necessary to equate the coefficients of the two lowest powers to zero individually. Therefore, a direct attack on the recurrence relation can be made at this point. For this purpose, the index is shifted on the first two sums and the resulting equation becomes

$$\sum_{n=0}^{\infty} \left[(n + c + 1)(2n + 2c + 1)a_{n+1} - a_n\right]z^{n+c} = 0$$

The recurrence relation is

$$a_{n+1} = \frac{a_n}{(n + c + 1)(2n + 2c + 1)} \qquad n \geq 0$$

or

$$a_n = \frac{a_{n-1}}{(n + c)(2n + 2c - 1)} \qquad n \geq 1 \tag{12b}$$

This recurrence relation can be used for both solutions to the indicial equation. However, the symbol for the coefficients should be changed in the second solution to avoid confusion.

The allowed values of c are 0 and $1/2$. They are substituted one at a time into equation (12b) to obtain the two linearly independent solutions to equation (12). When $c = 0$ is used,

$$a_n = \frac{a_{n-1}}{n(2n - 1)}$$

and the general term becomes

$$a_n = \frac{a_0}{n![1 \times 3 \times 5 \ . \ . \ . \ (2n - 1)]} = \frac{2^n}{(2n)!} a_0$$

Therefore,

$$w_0(z) = a_0 \sum_{n=0}^{\infty} \frac{2^n}{(2n)!} z^n$$

For the second solution ($c = 1/2$), it will be less confusing if the symbol for the coefficient of $z^{n+(1/2)}$ is denoted by b_n instead of a_n. Then equation (12b) becomes

$$b_n = \frac{b_{n-1}}{\left(n + \frac{1}{2}\right)(2n)} = \frac{b_{n-1}}{n(2n + 1)}$$

The general term in this case becomes

$$b_n = \frac{2^n b_0}{(2n + 1)!}$$

Therefore,

$$w_1(z) = \sum_{n=0}^{\infty} \frac{2^n z^{n+(1/2)}}{(2n + 1)!}$$

There are a number of clues which distinguish $w_1(z)$ corresponding to $c = 0$ from a solution about an ordinary point. For differential equations which have two-term recurrence relations, the solutions about an ordinary point always break up into a series of even powers and another one of odd powers, there being two solutions of integral powers. In this case, $w_0(z)$ has all powers of z, both even and odd. Furthermore, of course, the second solution is a series of fractional powers of z, rather than integral powers as would be the case for a solution about an ordinary point.

Example 3: Solution about a regular singular point where $s = 0$. - If a slight change is made in the preceding example, the case $s = 0$ results. The differential equation is

$$zw'' + w' - zw = 0 \tag{13}$$

Clearly, $z = 0$ is a regular singular point of this differential equation; $p_0 = 1$ and $q_0 = 0$. Therefore, equation (7) is

$$c^2 = 0$$

In this case, $c = 0$ and 0 and $s = 0$. Only one solution about $z = 0$ exists in power series form for this differential equation. The procedure for obtaining this solution is similar to the procedure in example 2. Upon substituting $w_c = \sum_{n=0}^{\infty} a_n z^{n+c}$ into equation (13) and combining terms of like powers of z, the result is

$$\sum_{n=0}^{\infty} (n + c)^2 a_n z^{n+c-1} - \sum_{n=0}^{\infty} a_n z^{n+c+1} = 0$$

The index is now shifted in the first sum and the following recurrence relation is obtained:

$$a_{n+2} = \frac{a_n}{(n + c + 2)^2} \qquad n \geq 0$$

or

$$a_n = \frac{a_{n-2}}{(n + c)^2} \qquad n \geq 2$$

The coefficient a_1 has been required to be equal to zero by virtue of equating the coefficient of z^0 to zero, so that no odd values of n will appear in the solution. The allowed values of n can therefore be most readily obtained by setting $n = 2k$, and the general term becomes

$$a_{2k} = \frac{a_0}{\prod_{j=1}^{k} (2j + c)^2}$$

Therefore, the series solution to equation (13) is

$$w_c = a_0\left\{z^c + \sum_{k=1}^{\infty} \frac{z^{2k+c}}{\left[\prod_{j=1}^{k} (2j + c)\right]^2}\right\} \tag{13a}$$

Now, $w_0 = \left(w_c\right)_{c=0}$ or

$$w_0(z) = a_0\left[1 + \sum_{k=1}^{\infty} \frac{z^{2k}}{2^{2k}(k!)^2}\right] = a_0 \sum_{k=0}^{\infty} \frac{\left(\frac{z}{2}\right)^{2k}}{(k!)^2}$$

The second solution is

$$w_1 = \left(\frac{\partial w_c}{\partial c}\right)_{c=0}$$

The procedure is standard but somewhat lengthy. Since the methods to be shown in the subsequent sections would not be helpful in reducing the effort involved in finding the second solution, the procedure is not exhibited in this report. Nevertheless, the solution itself is shown so that the description of the solution as not being expressible in series form will be fully appreciated. Thus,

$$w_1 = w_0(z)\log z - a_0 \sum_{k=1}^{\infty} \frac{H_k\left(\frac{z}{2}\right)^{2k}}{(k!)^2} \tag{13b}$$

where

$$H_k \equiv \sum_{j=1}^{k} \frac{1}{j}$$

is the sum of the first k terms of the harmonic series. Both terms on the right side of equation (13b) diverge as $z \to 0$. However, the sum of the two terms remains finite.

Example 4: Solution about a regular singular point where s is a positive integer - lucky case. - Consider the differential equation

$$9z^2w'' + 3z(z^2 + 2)w' + (z^2 - 2)w = 0 \tag{14}$$

In standard form, the equation becomes

$$w'' + \frac{z^2 + 2}{3z} w' + \frac{z^2 - 2}{9z^2} w = 0$$

Therefore, $z = 0$ is a regular singular point with $p_0 = 2/3$ and $q_0 = -2/9$. Under these conditions, equation (7) is

$$c^2 + \left(\frac{2}{3} - 1\right) c - \frac{2}{9} = 0$$

Thus, $c = 2/3$ and $-1/3$ and $s = 1$.

The procedure when s is an integer is to try using the smaller of the solutions to the indicial equation, that is, c_2. The reason for trying c_2 is that it is then found that either the recurrence relation for the coefficient a_s is impossible or else it will be automatically satisfied, leaving a_0 and a_s arbitrary and yielding the general solution from the work involving c_2 alone. When the recurrence relation for a_s leads to a set of inconsistent conditions, no solution using c_2 is found. The inconsistent condition which occurs is the requirement that a_0 be 0 - contrary to the hypothesis that a_0 be the coefficient of the first nonvanishing term of the series and, therefore, must be arbitrary. In such a case, the only series solution to the differential equation is one involving c_1, rather than c_2. In this example, two solutions are found by using c_2. Substitute $w = \sum_{n=0}^{\infty} a_n z^{n-(1/3)}$ into equation (14):

$$\sum_{n=0}^{\infty} \left[9\left(n - \frac{1}{3}\right)\left(n - \frac{4}{3}\right) + 6\left(n - \frac{1}{3}\right) - 2\right] a_n z^{n-(1/3)} + \sum_{n=0}^{\infty} \left[3\left(n - \frac{1}{3}\right) + 1\right] a_n z^{n+(5/3)} = 0$$

or

$$\sum_{n=0}^{\infty} 9n(n - 1)a_n z^{n-(1/3)} + \sum_{n=0}^{\infty} 3na_n z^{n+(5/3)} = 0$$

By equating the coefficients of $z^{-1/3}$ and $z^{2/3}$ to zero, the following requirements result:

$$0 \times a_0 = 0$$

$$0 \times a_1 = 0$$

so that a_0 and a_1 are both arbitrary. Note that $s = 1$, so $a_s = a_1$ here. The recurrence relation is found to be

$$a_n = -\frac{n - 2}{3n(n - 1)} a_{n-2} \qquad n \geq 2$$

Therefore, in this case, $a_{2k} = 0$, $k \geq 1$, and

$$w_0(z) = a_0 z^{-1/3} \tag{14a}$$

The solution involving a_1 is obtained by letting $n = 2k + 1$ and using

$$a_{2k+1} = -\frac{2k - 1}{6k(2k + 1)} a_{2k-1}$$

The general term is then found to be

$$a_{2k+1} = \frac{(-1)^k}{6^k k!\,(2k + 1)} a_1 \qquad k \geq 1$$

The second solution, which is also a series solution, is then

$$w_1(z) = a_1 \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k+(2/3)}}{6^k(2k+1)k!}$$

It should be noted that the theorem guaranteeing a series solution about a regular singular point specifically refers to using c_1 in the solution. In the "lucky" case, the second series solution always involves c_1 since it is a solution whose lowest power of z is $c_2 + s$, which is c_1.

Example 5: Solution about a regular singular point where s is an integer - unlucky case. - A common differential equation which illustrates this case is Bessel's equation of index 2:

$$z^2w'' + zw' + (z^2 - 4)w = 0 \tag{15}$$

The usual analysis shows that $z = 0$ is a regular singular point and that $p_0 = 1$ and $q_0 = 4$, so that the indicial equation is

$$c^2 - 4 = 0$$

Here $c = \pm 2$ and $s = 4$. As in the previous example, $c_2 = -2$ is tried first. Therefore, $w = \sum_{n=0}^{\infty} a_n z^{n-2}$ is substituted into equation (15). After simplifying, the results can be written as

$$\sum_{n=0}^{\infty} n(n-4)a_n z^{n-2} + \sum_{n=0}^{\infty} a_n z^n = 0$$

Since $s = 4$, we will equate coefficients of successively higher powers of z to zero until we have come to a_4. The coefficient of z^{-2} is

$$0 \times a_0 = 0$$

so a_0 is arbitrary. The coefficient of z^{-1} is $-3a_1 = 0$, so that $a_1 = 0$. The coefficient of z^0 requires contributions from both sums now:

$$2(-2)a_2 + a_0 = 0$$

so $a_2 = a_0/4$. Next,

$$-3a_3 + a_1 = 0$$

so $a_3 = 0$. Finally, we come to the coefficient of z^2:

$$0 \times a_4 + a_2 = 0$$

This requires that $a_2 = 0$. But $a_2 = a_0/4$ from a previous requirement. Therefore, the only way to keep the complete set of requirements on the coefficients consistent is to set $a_0 = 0$ also. This requirement is the one which is inconsistent with the hypothesis that a_0 be arbitrary. The result is that no series solution using $c_2 = -2$ can be found.

The only series solution for Bessel's equation is a series in powers of z^{n+2}. The second solution is found by a method somewhat similar to the one for the $s = 0$ case, but the algebra is often still more involved. The solution is expressed as $w_2(z) = \left(\partial w_c/\partial c\right)_{c=c_2}$ and is also a logarithmic solution. Neither solution will be displayed explicitly in this report.

THETA-OPERATOR METHOD

Two procedures which are used in obtaining series solutions for differential equations are often lengthy. These are (1) finding the indical equation in solutions about regular singular points and (2) finding the recurrence relation. Admittedly, equation (7) does not require much effort compared to the usual "textbook" method of equating to zero the coefficients of the lowest power of $z - z_0$. However, the theta operator yields the indicial equation and the recurrence relation automatically. As will be seen from the examples, the theta operator greatly reduces the total effort needed to obtain series solutions.

Furthermore, the method is applicable to n^{th} order, not just to second-order linear ordinary differential equations. This method has not been shown in any widely used textbooks, although it has been known for many years. Recently, however, Brand has finally included a good section on it in his book on differential and difference equations (ref. 2). Since he does not include the "standard" method in his book, the reader cannot appreciate the method as much as the comparison shown herein will permit.

Definitions and Properties

The definition of the theta operator is

$$\theta \equiv z \frac{d}{dz} = zD \tag{16}$$

where $D \equiv d/dz$.

Since we are interested in n^{th}-order linear ordinary differential equations we must be able to express D^n in terms of θ. To begin with, we show that

$$\theta(\theta - 1) = z^2 D^2 \tag{17}$$

<u>Proof</u>:

$$\theta(\theta - 1) = z \frac{d}{dz}\left(z \frac{d}{dz} - 1\right) = z\left[\left(z \frac{d^2}{dz^2} + \frac{d}{dz}\right) - \frac{d}{dz}\right] = z^2 D^2 \qquad \text{Q.E.D.}$$

Next, the generalized factorial notation is introduced:

$$(a)_n = a(a + 1). \; . \; .(a + n - 1) = \prod_{j=0}^{n-1} (a + j) \tag{18}$$

In terms of the Γ function, $(a)_n = \Gamma(a + n)/\Gamma(a)$. By using equation (18), the following relation can be proved by induction:

$$z^n D^n = (-1)^n (-\theta)_n \tag{19}$$

<u>Proof</u>: The relation certainly holds for $n = 1$ (by definition). The second part of the proof, therefore, consists in showing that, if it holds for $n = j$, it <u>must</u> hold for $n = j + 1$, $j \geq 1$. Now, if equation (19) holds for $n = j$, then

$$z^j D^j = (-1)^j (-\theta)_j \tag{19a}$$

Operate on equation (19a) by $(-1)(-\theta + j)$. The right side becomes $(-1)^{j+1} \prod_{k=0}^{j} (-\theta + j)$, which equals $(-1)^{j+1}(-\theta)_{j+1}$. Next, the left side is examined:

$$\theta(z^jD^j) = zD(z^jD^j) = z(jz^{j-1}D^j + z^jD^{j+1}) = jz^jD^j + z^{j+1}D^{j+1}$$

Since $(-1)(-\theta + j) = (\theta - j)$, the left side becomes

$$(\theta-j)(z^jD^j) = jz^jD^j + z^{j+1}D^{j+1} - jz^jD^j = z^{j+1}D^{j+1}$$

Therefore, $z^{j+1}D^{j+1} = (-1)^{j+1}(-\theta)_{j+1}$, $j \geq 1$, and the second part of the proof is complete.

The superiority of the θ-operator to the D-operator arises from the effect of the former on z^n: $\theta z^n = nz^n$. Therefore, any polynomial expression in θ operating on z^n yields the same polynomial in $\underline{n}$ times z^n. This property is useful enough to be expressed as a numbered equation. Thus,

$$P(\theta)z^n = P(n)z^n \tag{20}$$

In equation (20), the left side indicates a differential operation on z^n, whereas the right side is an ordinary algebraic expression.

An additional property of the θ-operator should be mentioned. It is sometimes convenient to make a change of independent variable of the form $u = az^k$. After making such a change of variable, the operator $u\, d/du$ is more convenient to use than θ itself. If this new operator is denoted by φ, then

$$\varphi = \frac{\theta}{k} \tag{21}$$

Proof:

$$u\frac{d}{du} = az^k \frac{dz}{du}\frac{d}{dz} = az^k \frac{1}{\frac{du}{dz}}\frac{d}{dz} = az^k \frac{1}{kaz^{k-1}}\frac{d}{dz} = \frac{1}{k} z\frac{d}{dz} = \frac{\theta}{k}$$

Note that for the differential equation in φ-form, the new $\bar{s}$ is $1/k$ times the old one.

Application of Theta Operator to Series Solutions of n^{th}-Order Linear Ordinary Differential Equations

A general n^{th}-order linear ordinary differential equation is usually written in the form

$$\left[c_0(z)D^n + c_1(z)D^{n-1} + \ldots + c_n(z)\right]w = 0$$

We shall limit ourselves to differential equations for which all the coefficients $c_i(z)$ are, at most, polynomials in z. Then, by using equation (19) whenever it is needed, such a differential equation can always be put into θ-form. The result is

$$\left[A_0(\theta) + zA_1(\theta) + \ldots + z^m A_m(\theta)\right]w = 0 \tag{22}$$

where all the $A_i(\theta)$ are polynomial operators in θ.

In order to show how the θ-operator can be used to solve such a general differential equation, we assume that $z = 0$ is a regular singular point so that w is of the form $\sum_{n=0}^{\infty} a_n z^{n+c}$. Upon substituting this series for w in equation (22) and using property (20), there results the equation:

$$\sum_{n=0}^{\infty} a_n A_0(n + c)z^{n+c} + \sum_{n=0}^{\infty} a_n A_1(n + c)z^{n+c+1} + \ldots + \sum_{n=0}^{\infty} a_n A_m(n + c)z^{n+c+m} = 0 \tag{23}$$

Clearly, the smallest exponent of z is c; and, in order that a_0 be arbitrary, the following indicial equation is automatically obtained:

$$A_0(c) = 0 \tag{7a}$$

The recurrence relation will be obtained once the index has been shifted by an appropriate amount in each series after the first one. Upon shifting the indices so that the sum in each series is over z^{n+c}, equation (23) becomes

$$\sum_{n=m}^{\infty} \left[A_0(n + c)a_n + A_1(n - 1 + c)a_{n-1} + \ldots + A_m(n - m + c)a_{n-m}\right](z - z_0)^{n+c} = 0 \tag{24}$$

The recurrence relation (which is an n-term one) is therefore

$$A_0(n + c)a_n + A_1(n - 1 + c)a_{n-1} + \ldots + A_m(n - m + c)a_{n-m} = 0 \tag{25}$$

The preceding results were obtained for a solution about $z = 0$. If a solution about some $z_0 \neq 0$ is desired, the following simple modifications will make the procedure for $z_0 \neq 0$ equivalent to that for $z_0 = 0$.

A transformation to a new independent variable $z' = z - z_0$ is made. Then each polynomial coefficient $c_i(z)$ becomes a polynomial in z', that is, $c_i'(z')$. Finally, a new operator $\theta' = z'(d/dz')$ is defined. After all these changes are made, an equation whose form exactly parallels that of equation (22) results, with $A_i'(\theta')$ and z' playing the same roles in the new equation as $A_i(\theta)$ and z play in equation (22).

It is, therefore, clear that there is no loss of generality in using an expansion about $z_0 = 0$.

The following steps summarize the θ-operator method for obtaining series solutions to n^{th}-order linear ordinary differential equations:

(1) Write the differential equation so that equation (19) can be applied.

(2) Write the differential equation in θ-form (i.e., eq. (22)).

(3) If a solution about a regular singular point is desired, the indicial equation is $A_0(c) = 0$.

(4) Write the recurrence relation using equation (25).

Of course, the rest of the problem, such as obtaining an explicit formula for a_n, must be done as previously. Naturally, the analysis which determines the nature of the point about which the solution is to be taken must precede the application of the θ-operator method. The following examples allow a comparison to be made of the amount of effort necessary to obtain solutions by the θ-operator method and the "standard" method.

Examples

Example 1: $2z(d^2w/dz^2) + (dw/dz) - w = 0$. - It was shown in the previous section that $z = 0$ is a regular singular point, so the method can be applied immediately. In order to write this differential equation in θ-form (see eq. (19)), the equation must first be multiplied by z. The resulting equation is

$$2z^2 \frac{d^2w}{dz^2} + z \frac{dw}{dz} - zw = 0$$

Now the conversion to θ-form becomes trivial:

$$[2\theta(\theta - 1) + \theta - z]w = 0$$

or

$$[\theta(2\theta - 1) - z]w = 0$$

Here, $A_0(\theta) = \theta(2\theta - 1)$, and $A_1(\theta) = -1$. Therefore, the indicial equation is $c(2c - 1) = 0$, which, of course, has the same solutions as were obtained by the standard method of Frobenius.

The recurrence relation is obtained immediately. It is

$$(n + c)[2(n + c) - 1]a_n - a_{n-1} = 0$$

or

$$a_n = \frac{a_{n-1}}{(n + c)(2n + 2c - 1)} \qquad n \geq 1$$

This is the same as equation (12b), and the procedure is unchanged from the "standard" method from here on.

Example 2: $zw'' + w' - zw = 0$. - As in example 1, this differential equation must be multiplied by z before equation (19) can be applied. After this has been done, the result is

$$z^2w'' + zw' - z^2w = 0$$

or, in θ-form,

$$\left[\theta(\theta - 1) + \theta - z^2\right]w = 0$$

which simplifies to

$$(\theta^2 - z^2)w = 0$$

Therefore, $A_0(\theta) = \theta^2$, $A_1(\theta) = 0$, and $A_2(\theta) = -1$. The indicial equation is $c^2 = 0$, and the recurrence relation is $(n + c)^2 a_n - a_{n-2} = 0$. Since the only series solution for this case (s = 0) is for $c = 0$, the relation $a_n = a_{n-2}/n^2$, where $n \geq 2$, is obtained as before.

Example 3: $9z^2w'' + 3(z^2 + 2)zw' + (z^2 - 2)w = 0$. - This equation is ready to be converted to θ-form as it stands. The result is

$$\left[9\theta(\theta - 1) + 3(z^2 + 2)\theta + (z^2 - 2)\right]w = 0$$

which simplifies to

$$\left[9\theta^2 - 3\theta - 2 + z^2(3\theta + 1)\right]w = 0$$

Here, $A_0(\theta) = 9\theta^2 - 3\theta - 2 = (3\theta - 2)(3\theta + 1)$, $A_1(\theta) = 0$, and $A_2(\theta) = 3\theta + 1$. The indicial equation is $(3c - 2)(3c + 1) = 0$, which results in $s = 1$, as before. The recurrence relation is $[3(n + c) - 2][3(n + c) + 1]a_n + [3(n + c - 2) + 1]a_{n-2} = 0$, $n \geq 2$. The rule in this case is to try the $c = -1/3$ solution. This attempt leads to the recurrence relation

$$a_n = -\frac{3n - 1 - 6 + 1}{(3n - 1 - 2)(3n - 1 + 1)}\, a_{n-2} = -\frac{n - 2}{3n(n - 1)}\, a_{n-2} \qquad n \geq 2$$

which was obtained by the standard method of Frobenius.

Of course, in working out a case where s is an integer, the coefficients of powers of z must be separately equated to zero until the coefficient a_s has been shown to be arbitrary. There is no point in looking for a recurrence relation until after the existence of a series solution for $c = c_2$ has been established. The θ-operator method cannot eliminate this part of the problem, but putting the differential equation into the form of equation (23) makes it easier to carry it out. In this example, equation (23) would take on the special form

$$\sum_{n=0}^{\infty} 9n(n - 1)a_n z^{n-(1/3)} + \sum_{n=0}^{\infty} 3na_n z^{[n-(1/3)+2]} = 0$$

Therefore, the coefficients of the two lowest powers are readily obtained from the equations: $0 \times a_0 = 0$, and $0 \times a_1 = 0$. Once it has been shown that these two coefficients are arbitrary, the recurrence relation can be found with confidence that the solution for $c = c_2$ exists.

It is hoped that these three examples have demonstrated the advantages of using the θ-operator in obtaining series solutions to linear ordinary differential equations.

SOLUTIONS USING GENERALIZED HYPERGEOMETRIC FUNCTIONS

It turns out that, for most of the linear ordinary differential equations which have found extensive application, the coefficients of the general term z^{n+c} can be written without resorting to a recurrence relation at all. Furthermore, all the series solutions that exist for these differential equations can be written with their coefficients completely determined. This is so because all these functions satisfy differential equations which are special cases of a single differential equation (27).

Admittedly, there do exist some linear ordinary differential equations which are not of the form permitting this to be done, and, for these, the use of the theta operator is recommended to reduce the labor involved in obtaining series solutions. However, a great many differential equations in common use do lend themselves well to the method described in this section, and series solutions for these equations can be obtained with a minimum of effort.

Definitions and Properties of ${}_pF_q(z)$

Definition. - The definition of a generalized hypergeometric function as used herein is (see eq. (18))

$$
{}_pF_q(z) = \sum_{n=0}^{\infty} \left[\frac{\prod_{i=1}^{p} (a_i)_n}{\prod_{j=1}^{q} (b_j)_n} \right] \frac{z^n}{n!} \tag{26}
$$

The a_i and b_j are called numerator parameters and denominator parameters, respectively. It should be noted that the $n!$ in the denominator is not included as a parameter in the notation. Nevertheless, as will be shown later, its presence cannot be ignored in constructing additional series solutions.

Another notation in common use is more cumbersome than the left side of equation (26) but gives more explicit information. This alternate notation is

$$F(a_1, \ldots, a_p;\ b_1, \ldots, b_q;\ z)$$

In the alternate notation the subscripts p and q are optional. The "grammar" in this notation is important. The list of denominator parameters is set off by semi-colons to denote the separation of these parameters from the numerator parameters to the left and from the variable to the right.

When written out, the right side of equation (26) takes the form

$$1 + \frac{a_1 \times a_2 \times \ldots \times a_p}{b_1 \times b_2 \ldots \times b_q} z + \frac{a_1(a_1+1)a_2(a_2+1) \ldots a_p(a_p+1)}{b_1(b_1+1)b_2(b_2+1) \ldots b_q(b_q+1)} \frac{z^2}{2!} + \ldots +$$

From this form it is clear that if any $a_i = 0$, ${}_pF_q(z) \equiv 1$.

Convergence of ${}_pF_q(z)$. - There are p distinct factorial factors in the numerator and $q + 1$ such factors of n in the denominator (the $n!$ must be included in considerations involving convergence). Therefore, applying the ratio test to equation (26) shows that ${}_pF_q(z)$ converges everywhere if $p < q + 1$, converges only at $z = 0$ if $p > q + 1$, and converges inside the unit circle if $p = q + 1$. The ordinary hypergeometric function of Gauss is a ${}_2F_1(z)$, and so it converges in the region $|z| < 1$.

A few common examples of functions expressible as ${}_pF_q(z)$ are

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} = {}_0F_0(z)$$

$$(1+z)^k = \sum_{n=0}^{\infty} \frac{k!}{(k-n)!} \frac{z^n}{n!} = \sum_{n=0}^{\infty} (-k)_n \frac{(-z)^n}{n!} = {}_1F_0(-k;\ -;\ -z)$$

$$\ln(1+z) = \sum_{n=0}^{\infty} \frac{(-1)^n z^{n+1}}{n+1} = z \sum_{n=0}^{\infty} \frac{n!(-z)^n}{(n+1)n!} = z \sum_{n=0}^{\infty} \frac{(1)_n(1)_n}{(2)_n} \frac{(-z)^n}{n!} = z\,{}_2F_1(1, 1;\ 2;\ -z)$$

$$\sin^{-1}z = z\ {}_2F_1(1/2, 1/2;\ 3/2;\ z^2)$$

$$\tan^{-1}z = z\ {}_2F_1(1/2, 1/2;\ 3/2;\ -z^2)$$

$$J_n(z) = \frac{\left(\frac{z}{2}\right)^n}{\Gamma(n+1)}\ {}_0F_1(-;\ n+1;\ -z^2/4)$$

Application of Generalized Hypergeometric Functions to Solutions of Linear n^{th}-Order Differential Equations

The complete set of generalized hypergeometric functions is too general for the applications. Since it is desirable to have convergence over a region more extensive than the point $z = 0$, it is necessary that $p \le q + 1$. Furthermore, if any denominator parameter were equal to a negative integer, $-\mu$, then the $(\mu + 1)^{th}$ term in equation (26) would have a zero in the denominator. As shown by Rule I (p. 30), the same result will occur in a related ${}_pF_q$ if any two denominator parameters differ by an integer.

Therefore, we will limit our discussion to generalized hypergeometric functions which are subject to these restrictions. Rainville used the θ-operator to derive the following differential equation which has such generalized hypergeometric functions as solutions (ref. 3):

$$\left[\theta \prod_{j=1}^{q} (\theta + b_j + 1) - z \prod_{i=1}^{p} (\theta + a_i)\right] w = 0 \tag{27}$$

where b_j is not a negative integer and no two b's differ by an integer.

This differential equation is very convenient to use since the denominator parameters are unambiguously separated from the numerator parameters. The convenience of this form is due entirely to the fact that the differential equation is in θ-form; such convenience disappears when the differential equation is written in a conventional manner (see example 1, p. 32).

By applying to equation (27) the discussion in the two preceding main sections, the following facts can be written at once:

(1) The point $z = 0$ is a regular singular point.

(2) The indicial equation is

$$c \prod_{j=1}^{q} (c + b_j - 1) = 0$$

Therefore, the $q + 1$ solutions to the indicial equation are $c = 0$ (common to all ${}_pF_q$'s) and $c = 1 - b_j$, where $j = 1, \ldots, q$.

(3) $A_0(\theta) = \theta \prod_{j=1}^{q} (\theta + b_j - 1)$; $A_1(\theta) = - \prod_{i=1}^{p} (\theta + a_i)$

Therefore, the recurrence relation is a two-term one. Since, in this section, the symbol a_i is reserved for the i^{th} numerator parameter, let α_n be the coefficient of z^n in the series for $w_0(z)$. Then the recurrence relation takes the form

$$\alpha_n = \frac{\prod_{i=1}^{p} (n + c - 1 + a_i)}{(n + c) \prod_{j=1}^{q} (n + c - 1 + b_j)} \alpha_{n-1} \qquad n \geq 1$$

The linear ordinary differential equation (27) is of $(q + 1)^{st}$ order and, therefore, has $q + 1$ linearly independent solutions. One of these is

$$w_0(z) = {}_pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z) \tag{28}$$

The other q linearly independent solutions can be obtained from $w_0(z)$ by using the following rule:

Rule I: Add $1 - b_m$ to every parameter in the numerator and the denominator (including the $(1)_n$ disguised as $n!$), and multiply the resulting function by z^{1-b_m}.

Thus,

$$w_m(z) = z^{1-b_m} {}_pF_q(a_1 + 1 - b_m, \ldots, a_p + 1 - b_m; b_1 + 1 - b_m, \ldots, b_{m-1} + 1 - b_m, 1 + 1 - b_m, b_{m+1} + 1 - b_m, \ldots, b_q + 1 - b_m; z) \tag{28a}$$

The $n!$ thus becomes the m^{th} denominator parameter, $2 - b_m$. The new ${}_pF_q$ is of the proper form, since it also has an $n!$ in the denominator obtained from the parameter which was b_m in $w_0(z)$. This parameter becomes $b_m + 1 - b_m = 1$ in the new ${}_pF_q$.

Remarks Concerning Reduction of Second-Order Linear Ordinary Differential Equations to the Form of Equation (27)

It is recommended that the theta operator be used in all instances, so that it is assumed that the differential equation is already in the form of equation (22). The following remarks may then be helpful in reducing second-order linear ordinary differential equations to the form of equation (27):

Remark (1). - Equation (27) indicates that a differential equation whose solution is a ${}_pF_q$ has only a two-term recurrence relation. Therefore, unless the differential equation in form (22) has a two-term recurrence relation, it cannot be put into the form (27), and no solution of the form ${}_pF_q$ exists.

Remark (2). - If equation (22) has a two-term recurrence relation, it will be of the form $\left[A_0(\theta) + z^k A_k(\theta)\right]w = 0$. At this point, a change of independent variable $t = -z^k$ should be made. This transformation will result in a new equation: $[A_0(k\varphi) - tA_k(k\varphi)]w = 0$, where $\varphi = t\, d/dt$. Following this transformation, the equation should be divided by the coefficient of φ^2 in A_0 and the new polynomial should be factored, resulting in $(\varphi - c_1)(\varphi - c_2)$, where c_1 and c_2 are the solutions to the new indicial equation in which t is the independent variable. At this stage, a decision must be made as to whether a change of dependent variable will be useful.

Remark (3). - In making this decision, it may be noted that the theta operator has the following effect on a function $w = z^n v$:

$$\theta(z^n v) = z^n(n + \theta)v \tag{29}$$

This property may be used to transform any term $A_0(\theta)w$ of the form $(\theta - c_1)(\theta - c_2)w$ to $z^{c_1}A_0(\theta + c_1)v$, which is of the form $z^{c_1}\theta[\theta + (c_1 - c_2)]v$, where $w = z^{c_1}v$. It should be noted that s still equals $c_1 - c_2$ in the new differential equation having v as the dependent variable. Therefore, the following useful rule has been established:

Rule II: Given that $A_0(\theta)w = (\theta - c_1)(\theta - c_2)w$, then the transformation $w = z^{c_1}v$ will result in a differential equation in which this term will be replaced by $\theta(\theta + s)v$.

Rule II enables a simple analysis to be made of ${}_pF_q$ solutions in the $s = 0$ case. When $s = 0$, the rule shows that $A_0(\theta)w$ will be replaced by $\theta^2 v$. Therefore, if a ${}_pF_q$ is a solution to this differential equation, the (single) denominator parameter will have to be $b = 1$. Therefore, $b - 1 = 0$, and the application of Rule I shows that there will never be a "second" ${}_pF_q$ solution about a point for which $s = 0$.

If s is a positive integer, μ, then $A_0(\theta)w$ will transform to $\theta(\theta + \mu)v$, and b will equal $1 + \mu$. Therefore, if a ${}_pF_q$ solution exists, Rule I should apply for finding the second solution.

Remark (4). - The discussion in remark (3) has been limited to an examination of $A_0(\theta)$. It is certainly a necessary condition that $A_0(\theta)$ be of the proper form in order that ${}_pF_q$ be a solution to the differential equation. However, it is not a sufficient condition. It is also necessary for $A_1(\theta)$ to be of the proper form.

See examples 4 and 5 in the next section to appreciate how these remarks can be applied.

Examples

Example 1: The hypergeometric equation. - In order to gain some confidence in the method, it will first be applied to an example which is tailormade for it, the hypergeometric equation of Gauss:

$$z(1 - z)w'' + \left[\gamma - (\alpha + \beta + 1)z\right]w' - \alpha\beta w = 0$$

The equation must be multiplied by z in order to get it into the θ-form. The resulting equation is

$$(1 - z)z^2 w'' + \left[\gamma - (\alpha + \beta + 1)z\right]zw' - \alpha\beta zw = 0$$

In θ-form, the equation becomes

$$\left\{(1 - z)\theta(\theta - 1) + \left[\gamma - (\alpha + \beta + 1)z\right]\theta - \alpha\beta z\right\}w = 0$$

or in the form of equation (27)

$$\left[\theta(\theta + \gamma - 1) - z(\theta + \alpha)(\theta + \beta)\right]w = 0$$

Therefore, one solution to this differential equation is

$$w_0(z) = {}_2F_1(\alpha, \beta;\ \gamma;\ z) = \sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta)_n}{(\gamma)_n} \frac{z^n}{n!}$$

The second solution is obtained by adding $1 - \gamma$ to each parameter, including the 1 which represents $n!$, and multiplying the new ${}_2F_1$ by $z^{1-\gamma}$. The result is

$$w_1(z) = z^{1-\gamma}\ {}_2F_1(\alpha + 1 - \gamma, \beta + 1 - \gamma;\ 2 - \gamma;\ z)$$

Thus, the two linearly independent solutions can be written almost on sight with all the coefficients of z^n in the series completely determined.

Now that we have seen how to use the method in a tailormade case, some of the examples used in previous sections of this report will be examined next.

Example 2: $2zw'' + w' - w = 0$. - This differential equation in θ-form was shown to be

$$\left[\theta(2\theta - 1) - z\right]w = 0$$

In order to put this equation into the form of equation (27), the equation must be divided by 2. This will allow the parameters to be correctly determined. Finally, then the form $\left[\theta\left(\theta - \frac{1}{2}\right) - \frac{z}{2}\right]w = 0$ results. In this form, it is clear that the hypergeometric function which is a solution to the differential equation has one denominator parameter and no numerator parameters and that the independent variable must be $z/2$, rather than z. Therefore,

$$w_0(z) = {}_0F_1(-;\ 1/2;\ z/2)$$

Furthermore, the second solution is

$$w_1(z) = z^{1-(1/2)}\ {}_0F_1(-;\ 1 + 1 - 1/2;\ z/2) = z^{1/2}\ {}_0F_1(-;\ 3/2;\ z/2)$$

The solutions in this form have completely determined coefficients and are certainly adequate for any computations which may be desired. In our case, however, we may feel more comfortable if we show explicitly that these are, indeed, the same solutions that were obtained previously. This task is readily accomplished with the use of the following relation which is listed in the next main section:

$$\left(\frac{a}{2}\right)_n = \frac{(a)_{2n}}{2^{2n}\left(\frac{a+1}{2}\right)_n}$$

For instance, in series form,

$$w_0(z) = \sum_{n=0}^{\infty} \frac{\left(\frac{z}{2}\right)^n}{n!\left(\frac{1}{2}\right)_n} = \sum_{n=0}^{\infty} \frac{z^n}{2^n\left(\frac{1}{2}\right)_n}$$

When $a = 1$,

$$\left(\frac{1}{2}\right)_n = \frac{(1)_{2n}}{2^{2n}\left(\frac{2}{2}\right)_n} = \frac{(2n)!}{2^{2n}n!}$$

If this is substituted into the series for $w_0(z)$, the result is

$$w_0(z) = \sum_{n=0}^{\infty} \frac{2^n \cancel{n!}}{(2n)!} \frac{z^n}{\cancel{n!}}$$

which agrees exactly with what was obtained by the standard method of Frobenius.

Similarly,

$$w_1(z) = z^{1/2} \sum_{n=0}^{\infty} \frac{\left(\frac{z}{2}\right)^n}{\left(\frac{3}{2}\right)_n n!} = \sum_{n=0}^{\infty} \frac{z^{n+(1/2)}}{2^n\left(\frac{3}{2}\right)_n n!}$$

Now,

$$\left(\frac{3}{2}\right)_n = \frac{\left(\frac{1}{2}\right)_n\left(\frac{3}{2} + n - 1\right)}{\frac{1}{2}} = (2n + 1)\left(\frac{1}{2}\right)_n = \frac{(2n + 1)!}{2^{2n} n!}$$

Therefore,

$$\frac{1}{2^n n!\left(\frac{3}{2}\right)_n} = \frac{2^n}{(2n + 1)!}$$

and $w_1(z)$ also agrees with the result obtained by the standard method.

Example 3: the $s = 0$ case (case b, p. 9). - The example of case b, for which $s = 0$ can be used to illustrate a second-order linear ordinary differential equation which has only one ${}_pF_q$ as a solution. In θ-form, the equation is

$$(\theta^2 - z^2)w = 0$$

Since this is an $s = 0$ case, only one series solution will exist. As recommended in remark (3), the change of independent variable $t = z^2$ should be made, thus transforming the differential equation to $(4\varphi^2 - t)w = 0$, or, upon dividing by 4, $(\varphi^2 - t/4)w = 0$.

This equation is in the form of equation (27), and the first solution is a ${}_0F_1$ with $t/4$ as the variable:

$$w_0(z) = {}_0F_1\left(-;\ 1;\ \frac{t}{4}\right) = \sum_{n=0}^{\infty} \frac{1}{(1)_n} \frac{\left(\frac{t}{4}\right)^n}{n!} = \sum_{n=0}^{\infty} \frac{t^n}{4^n (n!)^2} = \sum_{n=0}^{\infty} \frac{z^{2n}}{2^{2n}(n!)^2}$$

This is the same solution as was obtained by the standard method of Frobenius. However, the rule for obtaining the second solution fails here, since $b - 1 = 1 - 1 = 0$, so the "second" solution is the same as the first. Even hypergeometric functions cannot give a second series solution where none exists!

Example 4: the lucky s = integer case (case c, p. 10). - This differential equation in conventional notation was $9z^2w'' + 3(z^2 + 2)zw' + (z^2 - 2)w = 0$. In the section THETA-OPERATOR METHOD, it was shown that it could be written in θ-form as follows:

$$\left[(3\theta - 2)(3\theta + 1) + z^2(3\theta + 1)\right]w = 0$$

The change of independent variable $t = -z^2$ is indicated and results in the transformed differential equation: $(6\varphi - 2)(6\varphi + 1) - t(6\varphi + 1)\ w = 0$. Upon dividing by the coefficient of φ^2, this equation becomes

$$\left[\left(\varphi - \frac{1}{3}\right)\left(\varphi + \frac{1}{6}\right) - \frac{t}{6}\left(\varphi + \frac{1}{6}\right)\right]w = 0$$

An application of rule II is indicated. Therefore, the dependent variable is changed by using $w = t^{1/3}v$ to result in the differential equation

$$\left[\varphi\left(\varphi + \frac{1}{2}\right) - \frac{t}{6}\left(\varphi + \frac{1}{2}\right)\right]v = 0$$

This equation has a solution of the form

$$v_0(t) = {}_1F_1(1/2;\ 3/2;\ t/6)$$

from which one may obtain

$$w_0(z) = t^{1/3}v_0(t) = \left(-z^2\right)^{1/3}\ {}_1F_1(1/2;\ 3/2;\ -z^2/6) = (-1)^{1/3}z^{2/3}\ {}_1F_1(1/2;\ 3/2;\ -z^2/6)$$

This is the solution which was obtained as $w_1(z)$ by the standard method of Frobenius. To reconcile the forms, we may use equation (26)

$$z^{2/3}\ {}_1F_1(1/2;\ 3/2;\ -z^2/6) = z^{2/3}\sum_{n=0}^{\infty}\frac{\left(\frac{1}{2}\right)_n}{\left(\frac{3}{2}\right)_n}\frac{\left(-\frac{z^2}{6}\right)^n}{n!}$$

Using the property $(a + 1)_n = \left[(a + n)/a \; (a)_n\right]$ given in the compendium (p. 40), then, if $a = 1/2$,

$$\left(\frac{3}{2}\right)_n = \left[\left(\frac{1}{2}\right) + 1\right]_n = \frac{n + \frac{1}{2}}{\frac{1}{2}}\left(\frac{1}{2}\right)_n = (2n + 1)\left(\frac{1}{2}\right)_n$$

Therefore, $(1/2)_n \big/ (3/2)_n = 1/(2n + 1)$. Also, $\left(-z^2/6\right)^n = (-1)^n z^{2n} \big/ 6^n$. These changes convert the solution obtained here to the exact form of the solution obtained by the standard method of Frobenius.

If Rule I is applied to find the second solution to the differential equation here, the result is

$$v_1(t) = t^{-1/2} \; {}_1F_1(0;\; 1/2;\; -z^2/6) = t^{-1/2} \times 1$$

according to the note following the written-out series from equation (26). Therefore,

$$w_1(z) = t^{1/3} t^{-1/2} = t^{-1/6} = \left(-z^2\right)^{-1/6} = (-1)^{-1/6} z^{-1/3}$$

which agrees with $w_0(z)$ in the section STANDARD METHOD OF FROBENIUS.

Example 5: the unlucky s = integer case - Bessel's equation. - Consider Bessel's equation of order n, where n is a positive integer:

$$z^2 w'' + z w' + (z^2 - n^2) w = 0$$

In θ-form, this equation becomes

$$(\theta^2 - n^2 + z^2) w = 0$$

This is the case where $s = 2n$ and is a positive integer. Following the recommended procedure in remark (3), the change of independent variable $t = -z^2$ converts this equation to

$$(4\varphi^2 - n^2 - t) w = 0$$

which, upon dividing by 4, becomes

$$\left(\varphi^2 - \frac{n^2}{4} - \frac{t}{4}\right)w = 0$$

For this equation, $c_1 = n/2$, $c_2 = -n/2$ and $s = n$. Next, the change of dependent variable, $w = t^{n/2}v$, is made. The differential equation with v as the dependent variable is

$$\left[\varphi(\varphi + n) - \frac{t}{4}\right]v = 0$$

This equation is of the form (27) with the denominator parameter $b = n + 1$. Therefore, $v_0(t) = {}_0F_1(-; n + 1; t/4)$. From this solution,

$$w_0(z) = t^{n/2}v = z^n \, {}_0F_1(-; n + 1; -z^2/4) = 2^n \Gamma(n + 1)J_n(z)$$

from the short list of common functions expressible as ${}_pF_q$'s. Thus, the series for $J_n(z)$ has been obtained in a relatively few lines. As we saw before, the second solution to Bessel's equation is a logarithmic one. Since the denominator parameter is $n + 1$, 1 minus this parameter equals $-n$, and the rule for obtaining the second solution would result in a ${}_0F_1$ with a denominator parameter of $1 - n$. Since denominator parameters are not allowed to be zero or negative integers, a second solution in terms of hypergeometric functions cannot be obtained for integral n. It should also be noted that if $n = 0$, then $b = 1$ and the second solution is the same as the first (this is an $s = 0$ case).

It is hoped that these examples are sufficient to show the use of the theta operator and hypergeometric functions in obtaining series solutions to n^{th}-order linear differential equations.

SUMMARY OF PERTINENT APPROACHES TO OBTAINING SERIES SOLUTIONS OF LINEAR ORDINARY DIFFERENTIAL EQUATIONS

The first step in obtaining a series solution to a linear ordinary differential equation is an analysis of the point about which the series is to be expanded (see section SOME DEFINITIONS AND THEOREMS FOR SECOND-ORDER EQUATIONS). Such an analysis gives information concerning the number and kinds of series solutions which exist about that point. Following this analysis, it is recommended that the differential equation be put into theta form. At that point an attempt to put the equation in the form (27) may be made. If this attempt is successful (and the b_j's satisfy the restrictions preceding equation (27)), all $q + 1$ solutions may be written with the coefficients completely determined.

In those relatively rare cases in which the solutions cannot be (readily) expressed as ${}_pF_q$'s, the indicial equation and recurrence relation can be written for whatever solutions are expressible in series form.

The following section is a compendium of the formulae and properties of theta operators and generalized hypergeometric functions. It should be complete enough to enable these procedures to be followed without reference to external sources. There will probably be exceptions to this, but the author believes them to be rare.

COMPENDIUM OF PROPERTIES OF THETA OPERATOR AND GENERALIZED HYPERGEOMETRIC FUNCTIONS

Generalized Factorial Notation

$$(a)_n = \prod_{j=0}^{n-1} (a + j) = \frac{\Gamma(a + n)}{\Gamma(a)} \qquad \text{(see eq. (18))}$$

$$(a)_{n+1} = (a + n)(a)_n$$

$$(a + 1)_n = \frac{(a)_{n+1}}{a} = \frac{a + n}{a} (a)_n$$

$$\left(\frac{a}{2}\right)_n = \frac{(a)_{2n}}{2^{2n}\left(\frac{a + 1}{2}\right)_n}$$

For the special case where $a = 1$,

$$\left(\frac{1}{2}\right)_n = \frac{(1)_{2n}}{2^{2n}(1)_n} = \frac{(2n)!}{2^{2n}n!} = \frac{1 \times 3 \times 5 \ . \ . \ . \ (2n - 1)}{2^n}$$

This formula yields a simple expression for the product of odd integers - a fairly common occurrence.

Theta Operator

The theta operator is defined

$$\theta \equiv z \frac{d}{dz} \qquad \text{(see eq. (16))}$$

$$z^n D^n = (-1)^n (-\theta)_n \qquad \text{(see eq. (19))}$$

If $P(\theta)$ is a polynomial in θ, then

$$P(\theta)z^n = P(n)z^n \qquad \text{(see eq. (20))}$$

An n^{th}-order linear ordinary differential equation in theta form is

$$\left[A_0(\theta) + zA_1(\theta) + \ . \ . \ . \ + z^m A_m(\theta)\right]w = 0 \qquad \text{(see eq. (22))}$$

The indicial equation is

$$A_0(c) = 0 \qquad \text{(see eq. (7a))}$$

The recurrence relation is

$$A_0(n + c)a_n + A_1(n - 1 + c)a_{n-1} + \ . \ . \ . \ + A_m(n - m + c)a_{n-m} = 0 \qquad \text{(see eq. (25))}$$

For a change of independent variable - if $t = az^k$, then $\theta = k\varphi$ (see eq. (21)), where $\varphi = t\ d/dt$. In the differential equation in φ-form, the solutions to the indicial equation are c_1/k and c_2/k.

For a change of dependent variable,

$$\theta(z^n v) = z^n(\theta + n)v \qquad \text{(see eq. (29))}$$

Therefore,

$$(\theta - c_1)(\theta - c_2)w \rightarrow \theta(\theta + s)v \qquad \text{(see Rule II)}$$

where $w = z^{c_1}v$.

Generalized Hypergeometric Functions

The definition and notation is

$$ {}_pF_q(z) = F(a_1, \ldots, a_p;\ b_1, \ldots, b_q;\ z) = \sum_{n=0}^{\infty} \left[\frac{\prod_{i=1}^{p} (a_i)_n}{\prod_{j=1}^{q} (b_j)_n} \right] \frac{z^n}{n!} \qquad \text{(see eq. (26))} $$

The differential equation for ${}_pF_q$ written in theta form is

$$ \left[\theta \prod_{j=1}^{q} (\theta + b_j - 1) - z \prod_{i=1}^{p} (\theta + a_i) \right] w = 0 \qquad \text{(see eq. (27))} $$

The $q + 1$ solutions to this equation are

$$ w_0(z) = {}_pF_q(a_1, \ldots, a_p;\ b_1, \ldots, b_q;\ z) $$

$$ w_m(z) = z^{1-b_m}\, {}_pF_q(a_1 + 1 - b_m, \ldots, a_p + 1 - b_m;\ b_1 + 1 - b_m, \ldots, b_{m-1} + 1 - b_m, 2 - b_m, b_{m+1} + 1 - b_m, \ldots, b_q + 1 - b_m;\ z) \qquad m = 1, \ldots, q \qquad \text{(see eq. (28a))} $$

Lewis Research Center,
National Aeronautics and Space Administration,
Cleveland, Ohio, August 7, 1972,
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